ITTC Quality System Manual
Recommended Procedures and Guidelines

Procedure

Single Significant Amplitude and Confidence Intervals for Stochastic Processes

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Disclaimer

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Single Significant Amplitude and Confidence Intervals for Stochastic Processes

1. PURPOSE OF PROCEDURE

The purpose of this procedure is to formulate the process for characterizing the uncertainty of data resulting from a stochastic process, such as ship motion data in irregular seas, collected at either model scale, full scale or from numerical simulations. Ship motion data is collected in both controlled and uncontrolled environments—typically in a controlled environment for model-scale data and in an uncontrolled environment for full-scale data.

This procedure only deals with the statistical uncertainty of stochastic data resulting from the finite size of the sample. Although this procedure relates to all stochastic data, i.e. data resulting from the response to random excitation, it will be dealt with as though it is ship motion data. The uncertainty of significant wave height and modal period is presented in ITTC Recommended Procedure 7.5-02-07-01.4.

2. INTRODUCTION

Ship motion experimental data are considered random numbers because the environment is intrinsically random and the sample sizes are finite.

Ship motion data consists of time histories of ship motions including surge, sway, heave, roll, pitch, yaw, lateral and vertical accelerations at various locations on the vessel. etc. The data will consist of multiple records representing different model-test runs, full-scale measurement periods and computational durations. Ship motions are quantified as the mean, variance and single significant amplitude (SSA) of these quantities. Statistical uncertainty is expressed in terms of confidence intervals for the mean, variance and SSA estimates.

Statistical uncertainty is a result of the finite, rather than infinite, size of the sample data set, making averages random. The assumption of a normal distribution for these averages is based on the Central Limit Theorem.

A normal distribution is defined by its mean value and variance. The mean value of the estimate approximately equals the estimate itself. The variance of the estimate is computed from the time-series data. The uncertainty is quantified by the variance of the estimates (e.g. mean, variance and SSA).

The calculation of the variance of an estimate must account for the dependency of data points within each record that are close to each other in time.

Two different approaches are provided. One is for a case with a large number of records (runs) while the other is for *any* number of records.

3. FORMAT OF THE DATA

It is assumed that the process is stationary. In some cases, non-stationary data can be presented as stationary, see Recommended Procedure 7.5-02-01-06.

The sample of the process is represented as a set of independent records of different lengths, where dependence may be significant within each record. A nested array (array consisting of records of different length) is a convenient way to describe this type of data

$$X = \{X_j;\ j = 1, \dots, N_r\}$$
$$X_j = \{x_{j,i};\ i = 1, \dots, N_j\}, \quad (1)$$

where X is used to represent the entire sample, the index j, identifies a record and the index i identifies a point within a record. N_r is the number of records available in the sample, and N_j is the number of data points in the jth record. X_j identifies the jth record. The sample X is also referred to as an ensemble and includes all records corresponding to a single condition.

4. STATISTICAL UNCERTAINTY OF MEAN AND VARIANCE ESTIMATES

The statistical uncertainty of mean and variance estimates is strongly affected by statistical dependence within the data. Statistical dependence within ship motions reflects the physical nature of the rigid-body motion of a vessel on the surface of a dense fluid. Several phenomena contribute to this dependence. First, waves themselves carry dependence because water is a dense fluid, and the motion of water possesses significant inertia. A ship, acting essentially as a filter transforms this dependence in terms of its response. Second, a ship has its own inertia, which also contributes to dependence. Third, there are hydrodynamic forces proportional to the accelerations, generally referred to as added masses, which also contribute to dependences. The fourth contributor is the hydrodynamic memory effect, which is a result of the radiation and diffraction of waves.

However, the practical quantification of statistical uncertainty may not always require accounting for the statistical dependence. If there are a large number of runs, say 30 or more at the same condition, an explicit accounting for dependence may not be necessary. The methodology is straight forward, the mean and variance of each run is calculated separately, and they are then treated as an ensemble to compute the uncertainty of the runs in total as an ensemble.

In the general case with any number of runs, the dependence within the data run must be accounted for explicitly via an estimation of the autocovariance function, this is most important for a small number of records. The latter helps to quantify how much dependence is in the data. It is done by estimating the decorrelation time. Values in the time series of a stochastic process can be treated as independent random variables if the time interval between them is greater than or equal to the decorrelation time.

Correlation is often used as a measure of dependence in practical calculations. Correlation reflects dependence in terms of the second statistical moments. If random numbers are independent, their correlation is zero. However, zero correlation does not guarantee independence, as the latter may still manifest itself in high-order moments. While such a case was encountered in the relationship between roll angles and roll rates (Belenky and Weems, 2019), it is not known if dependence without correlation affects the objectives of this procedure, so it is assumed that the absence of correlation can be used as a criterion for independence.

4.1 Large Number of Records

For cases where data is presented by 30 or more independent records, the following procedure is applicable. The procedure starts by calculating the mean value estimate for each record:

$$\hat{E}_j = \frac{1}{N_j}\sum_{i=1}^{N_j} x_{j,i}, \tag{2}$$

where $x_{j,i}$ is the measured value of the jth record corresponding to time i. $\hat{E}_a$, an "ensemble" or "population" mean value estimate for all runs corresponding to a condition, is computed as:

$$\hat{E}_a = \sum_{j=1}^{N_r} W_j \hat{E}_j. \tag{3}$$

where N_r is the total number of runs for the condition and W_j is the statistical weight of each run, based on the temporal length of each run:

$$W_j = \frac{T_j}{\sum_{k=1}^{Nr} T_k} = \frac{T_j}{T_a}, \tag{4}$$

where T_j is the temporal length of record j and T_a is the cumulative time for all the records. If all of the records have the same sampling rate (time interval between data points), these weights can be calculated as:

$$W_j = \frac{N_j}{\sum_{k=1}^{Nr} N_k} = \frac{N_j}{N_a}, \tag{5}$$

where N_j is the number of data points in each record and N_a is the total number of data points in all records.

The variance of the mean for the ensemble is estimated using the formula:

$$\hat{\mathbb{V}}(\hat{E}_a) = \sum_{j=1}^{N_r} W_j^2 \left(\hat{E}_j - \hat{E}_a\right)^2 \tag{6}$$

An estimate of the variance of the variance for the ensemble is calculated starting using an estimate of the variance for each record of the ensemble. This is calculated using the ensemble mean:

$$\hat{V}_j = \frac{1}{N_j - 1} \sum_{i=1}^{N_j} \left(x_{j,i} - \hat{E}_a\right)^2 \tag{7}$$

The next step is to calculate the ensemble-averaged variance:

$$\hat{V}_a = \sum_{j=1}^{N_r} W_j \hat{V}_j. \tag{8}$$

The variance of the variance estimate is computed as

$$\hat{\mathbb{V}}(\hat{V}_a) = \sum_{j=1}^{N_r} W_j^2 \left(\hat{V}_j - \hat{V}_a\right)^2 \tag{9}$$

The results of Eqs. (6) and (9) are carried forward for the assessment of uncertainty of the final result. A more detailed discussion of the calculation of the estimates and the derivation of these formulae can be found in Belenky, *et al.* (2013, 2015).

4.2 Any Number of Records

For cases where data is presented by any number of records and particularly by fewer than 30 independent records, the variance of the mean and variance estimates can be evaluated using the estimates of the auto- and cross-covariance functions (Levine et al. 2017; Glotzer et al. 2023), as well as the decorrelation time.

4.2.1 Estimation of Auto- and Crosscovariance Functions

The autocovariance function of the *j*th record is estimated as:

$$\hat{R}_k(X_j) = \frac{1}{N_j} \sum_{i=1}^{N_j - k} \left(x_{j,i} - \hat{E}_a\right) \times \left(x_{j,i+k} - \hat{E}_a\right), \tag{10}$$

where $k = 0,1,..N_j - 1$ is the index for the time lag. Note that the estimate of the autocovariance function computed for zero-time lag (k = 0) is the estimate of variance. Figure 1 illustrates the autocovariance function estimated from a single record of roll motions.

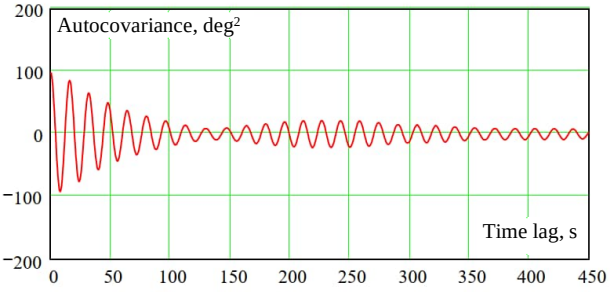

Figure 1: Estimate of autocovariance from a single record of roll motions

The estimate has an oscillatory character. Its magnitudes decrease until about 140 s and then

increases again. This increase of magnitudes is a result of insufficient data at larger time lags, as there are no physical reasons for further dependence as the time passes, while there are fewer data points available with every time lag. The decrease in the number of data points leads to an appearance of "statistical noise", manifesting itself in the increase of the magnitude of the autocovariance function. This "noise" has zero mean and its influence can be mitigated by averaging the autocovariance estimate over all available records.

The ensemble averaged estimate of the autocovariance function is expressed as:

$$\hat{R}_k(X) = \sum_{j=1}^{N_r} W_j \hat{R}_k(X_j). \quad (11)$$

The effect of ensemble averaging is illustrated in Figure 2. The magnitudes of ensemble-averaged estimate experiences less increase in comparison to the single-record estimates. The statistical "noise" from each individual record's estimate appears to cancel each other.

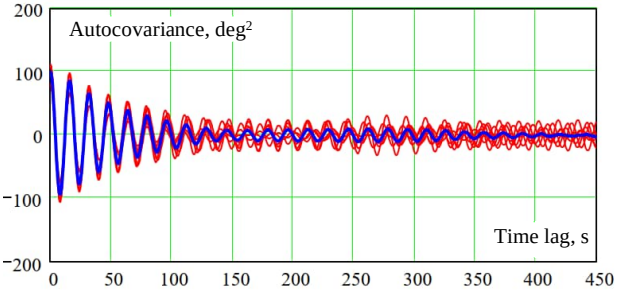

Figure 2: Estimation of the ensemble-averaged autocovariance function (blue). Record estimates are in red

The mean of the squares for the *j*th record is estimated as:

$$\hat{E}(X_j^2) = \frac{1}{N}\sum_{i=1}^{N_j} x_i^2 \quad (12)$$

and the ensemble-averaged estimate of mean of the squares is

$$\hat{E}(X^2) = \sum_{j=1}^{N_r} W_j \hat{E}(X_j^2)\,. \quad (13)$$

The estimate of autocovariance of the squares is computed in a manner similar to that in Eqs. (10) and (11):

$$\hat{R}_k(X_j^2) = \frac{1}{N_j}\sum_{i=1}^{N_j-k}\left(x_{j,i}^2 - \hat{E}(X^2)\right) \times \left(x_{j,i+k}^2 - \hat{E}(X^2)\right), \quad (14)$$

$$\hat{R}_k(X^2) = \sum_{j=1}^{N_r} W_j \hat{R}_k(X_j^2). \quad (15)$$

Finally the covariance function for the process and its squares for each record of the process and its ensemble is estimated as:

$$\hat{C}_k(X_j, X_j^2) = \frac{1}{N_j}\sum_{i=1}^{N_j-k}(x_{j,i} - \hat{E}_a) \times \left(x_{j,i+k}^2 - \hat{E}(X^2)\right), \quad (16)$$

$$\hat{C}_k(X, X^2) = \sum_{j=1}^{N_r} W_j \hat{C}_k(X_j, X_j^2). \quad (17)$$

Estimates of the autocovariance function of the squares and covariance function for the process and its squares are given in Appendix B.

4.2.2 Evaluation of Decorrelation Time

The autocorrelation function is the autocovariance function normalized by the first term (k = 0; it is the variance estimate).

$$\hat{r}_k(X) = \frac{\hat{R}_k(X)}{\hat{R}_0(X)} = \frac{\hat{R}_k(X)}{\hat{V}_a} \quad (18)$$

The appearance of the estimate of the autocorrelation function for roll motions is similar to the autocovariance function shown in Figure 3.

The decorrelation time τ_{dc} is estimated from the ensemble-averaged estimate of the autocorrelation function Eq. (11).

As the autocorrelation function of ship motions is oscillatory, the decorrelation time is estimated using its envelope. In Figure 3, the envelope of the autocorrelation function is plotted

by connecting the absolute values of the peaks of the autocorrelation function by a blue line.

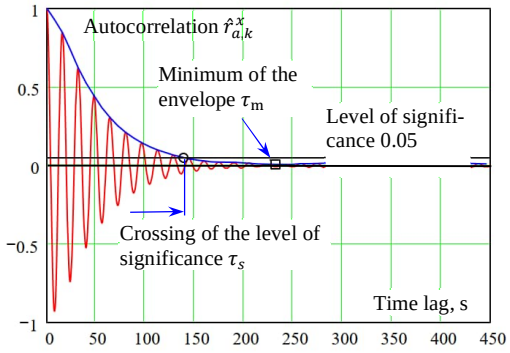

Figure 3: Estimate of decorrelation time with ensemble-average autocovariance function

Two quantities are computed with the envelope of the autocorrelation estimate. The first quantity is the time lag at which the envelopes crosses the levels of significance (taken as 0.05) for the first time (τ_s), see Figure 3. The second quantity is the position of the first local minimum of the envelope (τ_m) also shown in Figure 3. The decorrelation time is taken as the smaller of these two quantities:

$$\tau_{dc} = \min(\tau_s, \tau_m). \tag{19}$$

The justification is as follows. In most practical cases, a correlation below the significant level is taken as an absence of correlation. However, the available sample volume may be insufficient to mitigate the influence of "statistical noise" enough to reduce the envelope below the level of significance.

The first local minimum of the envelope may be used as an indicator of the start of "statistical noise". The correlation is expected to subside with time as there are no physical reasons for increased dependence appearing later in the process. Thus, the first local minimum can be interpreted as the longest time for which meaningful correlation was observed. The case shown in Figure 3 has enough data that crossing the level of significance is used to estimate the decorrelation time. A case in which the minimum of the envelope is used to estimate the decorrelation time is illustrated in Appendix B.

4.2.3 Calculating Variances of the Mean and Variance Estimates

The variance of the mean estimate for the *j*th record is computed from the autocovariance function of the process:

$$\begin{aligned}\hat{\mathbb{V}}(\hat{E}_j) &= \frac{\hat{V}_a}{N_j} + \frac{2}{N_j}\sum_{k=1}^{M}\left(1-\frac{k}{N_j}\right)\hat{R}_k(X)F_k \\ &= \frac{1}{N_j}\sum_{k=-M}^{M}\left(1-\frac{|k|}{N_j}\right)\hat{R}_{|k|}(X)\,F_{|k|}\end{aligned} \tag{20}$$

where M is the index corresponding to the decorrelation time τ_{dc}, $F_k = 1 - k/M$ is a windowing function, used to eliminate the "statistical noise", remaining in the ensemble-averaged estimate of the autocovariance function.

The variance of the ensemble-averaged mean estimate is computed as:

$$\hat{\mathbb{V}}(\hat{E}_a) = \sum_{j=1}^{N_r} W_j^2\,\hat{\mathbb{V}}(\hat{E}_j). \tag{21}$$

The following quantities needs to be computed to estimate the variance of the variance. The variance of the mean estimate of the squares of the *j*th record is:

$$\begin{aligned}\hat{\mathbb{V}}\left(\hat{E}(X_j^2)\right) &= \frac{1}{N_j}\sum_{k=-\mathrm{M}}^{M}\left(1-\frac{|k|}{N_j}\right) \\ &\quad\times \hat{R}_{|k|}(X^2)F_{|k|}.\end{aligned} \tag{22}$$

The ensemble averaged value is

$$\hat{\mathbb{V}}\left(\hat{E}(X^2)\right) = \sum_{j=1}^{N_r} W_j^2\,\hat{\mathbb{V}}\left(\hat{E}(X_j^2)\right) \tag{23}$$

To reflect the dependence between the process and its squares (which may be non-zero for a non-Gaussian process), the covariance of the *j*th record is estimated as:

$$\hat{\mathbb{C}}\left(\hat{E}_j, \hat{E}\left(X_j^2\right)\right) = \frac{1}{N_j}\sum_{k=-\mathrm{M}}^{M}\left(1-\frac{|k|}{N_j}\right) \times \hat{C}_{|k|}(X, X^2)F_{|k|}. \quad (24)$$

The ensemble average value is computed as:

$$\hat{\mathbb{C}}\left(\hat{E}_a, \hat{E}(X^2)\right) = \sum_{j=1}^{N_r} W_j^2\, \hat{\mathbb{C}}\left(\hat{E}_j, \hat{E}\left(X_j^2\right)\right). \quad (25)$$

Finally, the variance of the variance estimate for all of the data is evaluated as:

$$\hat{\mathbb{V}}(\hat{V}_a) = \hat{\mathbb{V}}\left(\hat{E}(X^2)\right) + 4\left(\hat{E}_a\right)^2\hat{\mathbb{V}}(\hat{E}_a) - 4\hat{E}_a\hat{\mathbb{C}}\left(\hat{E}_a, \hat{E}(X^2)\right). \quad (27)$$

If it is necessary to evaluate the variance of the variance estimate of an individual record, Eq. (27) can be applied to a single record's values as defined by Eqs. (20), (22), and (24).

5. SINGLE SIGNIFICANT AMPLITUDE ESTIMATE AND ITS STATISTICAL UNCERTAINTY

By definition, the Single Significant Amplitude (SSA) is the average of the largest 1/3 of the peaks of the stochastic process with respect to its mean. If the process is Gaussian, its SSA can be expressed using its variance

$$\widehat{SSA}_X = 2\sqrt{\hat{V}_X}. \quad (28)$$

If Eq. (28) is used without the process being known to be normal, it can be considered to be a convenient approximation for the variance and/or standard deviation estimates.

The application of this approximation is convenient for quick analysis. The SSA value can be directly estimated from the data without any knowledge regarding its probability distribution.

5.1 Estimation of Single Significant Amplitude

Using the ensemble mean estimate Eq. (3), the mean-crossing points (times) are found. These points are presented as a nested array

$$TM = \{TM_j;\ j = 1, \ldots, N_r\}$$
$$TM_j = \{Tm_{j,m};\ m = 1, \ldots, Nm_j\}, \quad (29)$$

where m is the index of mean crossings and Nm_j is the number of mean crossings in the jth record.

The next step is to search for the peak values, which are defined as the largest distance of the absolute value of the sample from the mean between each pair of consecutive crossings of the level of the mean value estimate, see Figure 4.

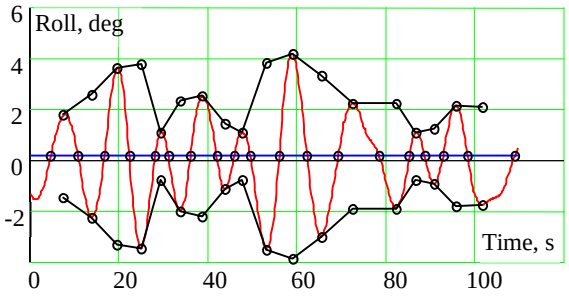

Figure 4: Definition of mean-crossing peaks

The total number of peak values will be:

$$N_p = \sum_{j=1}^{N_r}\left(Nm_j - 1\right) \quad (30)$$

The peak data nested array must also include the time corresponding to these points. The data may be presented in the form of a matrix with three columns — record index, time of peak value in the record, and the peak value — with N_p rows. The matrix is defined as follows:

$$\mathbf{Pk} = \begin{cases} Pk_{k,1} = j \\ Pk_{k,2} = tp_{j,m}\ ;\ k = 1, \ldots N_p. \\ Pk_{k,3} = xp_{j,m} \end{cases} \quad (31)$$

where k is the index of the peak value, which is advanced every time a mean-crossing peak is identified, the time and value of the peak are defined as:

$$xp_{j,m} = \max\left(|x_{j,i} - \hat{E}_a|\right) \text{ when } t_i \in \left[Tm_{j,m}, Tm_{j,m+1}\right] \quad (32)$$

and $tp_{j,m}$ corresponds to $xp_{j,m}$ while

$$k = 1, \ldots, Np \quad \text{and} \quad m = 1, \ldots, Nm_j - 1. \quad (33)$$

The next step is to find the level corresponding to the largest 1/3 of the peak values. It is, in fact, an estimation of the largest 1/3 quantile. This is found by sorting the peaks by value (highest to lowest) and finding the value of the peak that encompasses the largest 1/3 set of the sorted list of peaks:

$$\begin{aligned} \overrightarrow{Pks} &= \text{sort}(Pk^{<3>}) \\ id_{1/3} &= \text{round}\left(\frac{Np}{3}\right) \\ \hat{a}_{1/3} &= Pks_{ids} \end{aligned} \quad (34)$$

where $\mathbf{Pk}^{<3>}$ means the "3rd column of the matrix **Pk**".

The sample of the largest 1/3 of the peaks is then extracted from the matrix **Pk**:

$$\mathbf{Ps} = \begin{cases} Ps_{l,1} = Pk_{k,1} \\ Ps_{l.2} = Pk_{k,2} \text{ if } Pk_{k,3} > \hat{a}_{1/3} \\ Ps_{l,3} = Pk_{k,3} \end{cases} \quad (34)$$

where $l = 1, \ldots, Ns$; and Ns is the total number peaks above the 1/3-quantile estimate and will, by definition, be the same as the index *ids* above:

$$N_s = id_{1/3} = \text{round}\left(\frac{Np}{3}\right). \quad (36)$$

The estimate of SSA is the estimate of the mean value of $Ps^{<3>}$:

$$\widehat{SSA} = \hat{E}(Ps^{<3>}) = \frac{1}{N_s}\sum_{l=1}^{N_s} Ps_{l,3}. \quad (37)$$

Figure 5 illustrates the peak average calculation for a sample roll time history. Peak values are marked in green or blue. The level of the largest 1/3 of the peak values is set as the 66.6th percentile of all peaks and is shown as the green line on the figure. The peaks above this value are marked in blue and comprise the largest-1/3 of the peaks. The average of these blue peaks is the SSA estimate, which is shown as the blue line.

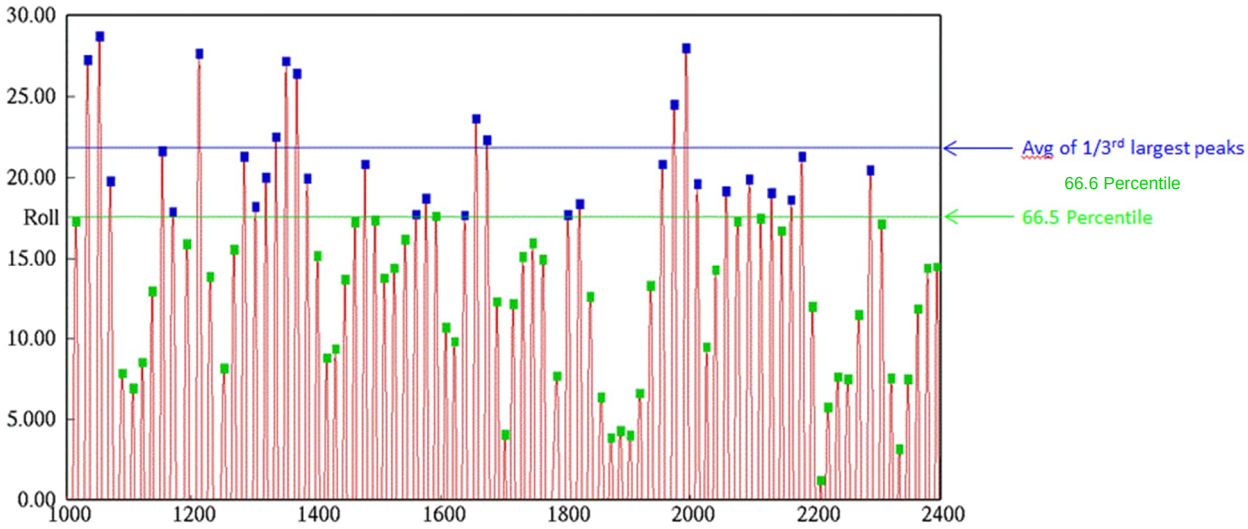

Figure 5: Peak average calculation

5.2 Uncertainty of Single Significant Amplitude Estimate

To find the variance of the SSA estimate, Eq. (37), the dependence within the sample $Ps^{<3>}$ must be addressed. The largest 1/3 of the peaks, as well as all of the peaks, are values of the motion process at particular instants of time, which may be dependent on one another if they are from the same record and are close together in time.

Peaks that were recorded within the time difference τ_{ds}, are assumed to be dependent. These peaks are grouped together. These groups comprise a series of successive peaks, which are from the same record and for which the time interval between peaks is less than τ_{dc} .

A single independent peak that is not a part of any other group makes its own group.

As these groups may consist of different number of peaks, a nested array is again a natural way to organize this data:

$$PC = \{PC_j;\ j = 1, \ldots, N_{pc}\}$$
$$PC_j = \{Pc_{j,i};\ i = 1, \ldots, Nc_j\}, \quad (38)$$

where Nc_j is the number of peaks in the jth "group" and N_{pc} is the number of "groups" found while analyzing the dependence between the largest 1/3 of the peaks.

Once the peaks have been sorted into groups, the variance of the peak average estimate can be computed using a procedure similar to that for the mean or variance of a sample without the assumption of large number of independent records.

The calculation for the variance of the peak average follows the mean or variance approach, except that:

- A group can contain a single peak
- Peaks from different groups are assumed to be independent.
- The auto-covariance with respect to the difference in peak index, rather than time lag,

is used to account for the dependence between peaks in a group
- Autocovariance is averaged over groups, rather than records

The autocovariance function for the largest 1/3 of the peaks is estimated as:

$$\hat{R}^p_{a,k} = \sum_{j=1}^{N_g} Wg_j \hat{R}^p_{j,k} \quad (39)$$

where $\hat{R}^p_{j,k}$ is the autocovariance estimate for each group of dependent peaks

$$\hat{R}^p_{j,k} = \begin{cases} \frac{1}{Nc_j}\sum_{i=1}^{Nc_j-k}\left(Pc_{j,i} - \widehat{SSA}\right) \\ \quad \times \left(Pc_{j,i+k} - \widehat{SSA}\right); & Nc_j < k \\ 0\,; & Nc_j \geq k \end{cases} \quad (41)$$

while Wg_j is the statistical weight of each group of dependent peaks:

$$Wg_j = \frac{Nc_j}{N_s}\,. \quad (42)$$

Figure 6 shows an example of the averaged autocorrelation for roll as estimated using Eq. (39). As can be seen, the dependence covers about 4–5 sequential peaks.

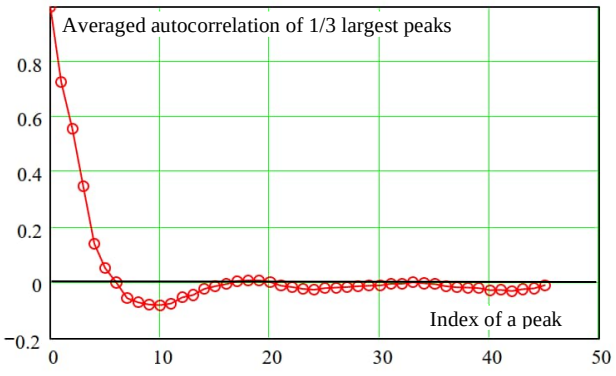

Figure 6: Estimate of averaged autocovariance for largest 1/3 of the peaks for roll motion

Similar to decorrelation time, a decorrelation peak index if found to determine how long the dependence lasts among the largest 1/3 of the peaks. The autocorrelation function of the largest 1/3 of the peaks does not have the same obvious oscillatory character that the autocorrelation function of ship motion does. The decorrelation index is estimated as the smallest number of peaks between either the crossing of the significance level or the first local minimum, see the example in Appendix B.

Once the averaged autocovariance function has been estimated using Eq. (39), the variance of the SSA estimate can be found as follows:

$$\hat{V}_{\widehat{SSA}} = \sum_{j=1}^{N_r} Wg_j^2 \hat{V}_{\widehat{SSA}j} \quad (43)$$

where

$$\hat{V}_{\widehat{SSA}j} = \frac{\hat{R}^p_{a,0}}{Nc_j} + \frac{2}{Nc_j}\sum_{k=1}^{M_p-1}\left(1 - \frac{k}{Nc_j}\right)\hat{R}^p_{a,k}F^p_k \quad (44)$$

$$F^p_k = 1 - \frac{k}{M_p} \quad (45)$$

M_p is the decorrelation index of the largest 1/3 of the peaks

6. CONSTRUCTING CONFIDENCE INTERVALS

6.1 General

For a sample estimate, the confidence interval (CI) is defined (which reflects its uncertainty, caused by its random nature). A two-sided CI is an interval around the estimate where the true value is contained with a given probability (confidence probability or confidence level).

The calculation of the confidence interval requires the assumption of the probability distribution of the estimate. The lower and upper boundaries S_{Low} and S_{Up} of the confidence interval of the estimate S are calculated with quantiles (inverse of the cumulative distribution

function). For two-sided confidence interval, these boundaries are:

$$S_{Low} = Q\left(\frac{1-P_\beta}{2}\right); \quad S_{Up} = Q\left(\frac{1+P_\beta}{2}\right) \tag{46}$$

where P_β is the accepted confidence probability and Q stands for quantile.

As stated in Section 2, the normal distribution is assumed for all estimates. For the normal distribution, the center of the range is determined by the calculated statistical estimate. The width of the range is determined from the variance of this estimate. Note that this does *not* assume that the distribution of the process itself is normal.

If a normal distribution of the estimate is assumed, Eq. (46) can be simplified, because the normal distribution is symmetric:

$$S_{Low} = S - K_\beta\sqrt{\mathrm{V}_S} \quad S_{Up} = S + K_\beta\sqrt{\mathrm{V}_S} \tag{47}$$

where V_S is the variance of the estimate S and the coefficient K_β is one-half the non-dimensional width of the confidence interval and is calculated as:

$$K_\beta = Q_N\left(\frac{1+P_\beta}{2}\right) \tag{48}$$

where Q_N is a quantile of the standard normal distribution (zero-mean & unity-variance). For the typical confidence probably of 0.95, K_β equals 1.96.

6.2 Confidence Interval for Mean Value and Variance Estimates

The upper and lower bounds of the confidence interval for the mean values are:

$$\hat{E}_{Low} = \hat{E}_a - K_\beta\sqrt{\hat{V}_{\hat{E}a}} \quad \hat{E}_{Up} = \hat{E}_a + K_\beta\sqrt{\hat{V}_{\hat{E}a}}. \tag{49}$$

In these expressions, $\hat{E}_a$ is the mean estimate for the ensemble, Eq. (3); and $\hat{V}_{\hat{E}a}$ is the estimate of the variance of the mean for the ensemble, Eq. (6) for a large number of runs or Eq. (21) for any number of runs.

Expressions for the confidence interval of the variance estimates are:

$$\hat{V}_{Low} = \hat{V}_a - K_\beta\sqrt{\hat{V}_{\hat{V}a}} \quad \hat{V}_{up} = \hat{V}_a + K_\beta\sqrt{\hat{V}_{\hat{V}a}} \tag{50}$$

where $\hat{V}_a$ is the variance estimate for the ensemble, calculated with Eq. (8); $\hat{V}_{\hat{V}a}$ is the variance of the variance estimate for the ensemble, calculated with Eq. (9) for a large number of runs or Eq. (27) for any number of runs.

6.3 Confidence Interval for SSA Estimate

If the SSA is estimated from the data as described in subsection 5.1, the upper and lower boundaries of the confidence interval are:

$$\widehat{SSA}_{Low} = \widehat{\mathrm{SSA}} - K_\beta\sqrt{\hat{V}_{\widehat{\mathrm{SSA}}}} \quad \widehat{SSA}_{Up} = \widehat{\mathrm{SSA}} + K_\beta\sqrt{\hat{V}_{\widehat{\mathrm{SSA}}}} \tag{51}$$

where the SSA estimate is computed with Eq. (37) and its variance $\hat{V}_{\widehat{\mathrm{SSA}}}$ is computed using Eq. (43).

For rapid approximate analysis, when Eq. (28) is used to estimate the SSA, the boundaries of its confidence interval can be computed using the boundaries of the variance estimate, defined by Eq. (50):

$$\widehat{SSA}_{Low} = 2\sqrt{\hat{V}_{Low}}$$
$$\widehat{SSA}_{Up} = 2\sqrt{\hat{V}_{Up}}. \quad (52)$$

Alternatively, the boundaries of the confidence interval can be computed using the Delta method, see Appendix A:

$$\widehat{SSA}_{Low} = 2\sqrt{\hat{V}_a} - K_\beta\sqrt{\frac{\hat{V}_{\hat{V}a}}{\hat{V}_a}}$$
$$\widehat{SSA}_{Up} = 2\sqrt{\hat{V}_a} + K_\beta\sqrt{\frac{\hat{V}_{\hat{V}a}}{\hat{V}_a}} \quad (53)$$

where $\hat{V}_a$ is the variance estimate for the ensemble, calculated with Eq. (8); $\hat{V}_{\hat{V}a}$ is the variance of the variance estimate for the ensemble, calculated using Eq. (9) for a large number of runs or Eq. (27) for any number of runs.

6.4 Alternative Method: Self-Normalization

Self-normalization is an alternative approach to constructing confidence intervals from the data. The background and references for this approach are available in Pipiras et al. (2018) and Glotzer et al. (2023). The method is simple to implement as it does not require the autocorrelation of the data, so the application of the method does not depend on the number of runs.

In this approach, the data are organized in a simple array:

$$Y = \{y_i;\ i = 1, \dots, N_y\} \quad (54)$$

where N_y is the total number of points available. First, a cumulative sum is computed:

$$ST_j = \sum_{i=1}^{j}(y_i - \hat{E}_y)\ j = 1, \dots, N_y, \quad (55)$$

where $\hat{E}_y$ is the mean value estimate of the data array *Y*. Then the sum of the squares of (55) is computed:

$$SQ = \frac{1}{N_y^2}\sum_{i=1}^{N_y} ST_i^2. \quad (56)$$

The boundaries of the confidence interval for the mean value estimate $\hat{E}_Y$ are computed as:

$$\hat{E}_{YLow} = \hat{E}_Y - u_\alpha\sqrt{\frac{SQ}{N_y}}$$
$$\hat{E}_{YUp} = \hat{E}_Y + u_\alpha\sqrt{\frac{SQ}{N_y}} \quad (57)$$

where u_α is the critical value. For the typical confidence probably of $P_\beta = 0.95$

$$u_\alpha = \sqrt{45.4} = 6.738. \quad (58)$$

Critical values for other confidence probabilities are available in Lobato (2001).

To construct the confidence interval for the mean value, Eqs. (55)–(57) are applied to time history data. To construct the confidence interval for the variance estimate, Eqs. (55)–(57) are applied to the centered squares of the data:

$$z_i = \left(y_i - \hat{E}_a\right)^2, i = 1, \dots, N_y \quad (59)$$

where y_i are time history data.

To construct the confidence interval for the SSA estimate, Eqs. (55)–(57) are applied to the largest 1/3 of the peaks:

$$y_l = Ps_{l,3}\ ;\ \ l = 1, \dots, Ns. \quad (60)$$

In many practical cases, the confidence interval constructed using self-normalization is slightly wider than that obtained using the other methods.

7. LIST OF SYMBOLS

$\hat{a}_{1/3}$ Process value corresponding to largest 1/3 of the peaks.
$\hat{\mathbb{C}}(\dots, \dots)$ Covariance.

$\hat{\mathbb{C}}\left(\hat{E}_a, \hat{E}(X^2)\right)$ Covariance between the ensemble-averaged mean estimate of the process and the the ensemble-averaged mean estimate of process of the squares.

$\hat{\mathbb{C}}\left(\hat{E}_j, \hat{E}(X_j^2)\right)$ Covariance between the mean estimate of the process and the mean estimate of process of the squares of *j*th record.

$\hat{C}_k(X_j, X_j^2)$ *k*th time lag of covariance function estimate of *j*th record of the process *X* and the process of the squares.

$\hat{C}_k(X, X^2)$ *k*th time lag of ensemble-averaged covariance function estimate of the process *X* and the process of the squares.

$\mathbb{E}(\dots)$ Expected value

$\hat{E}_a$ Ensemble-averaged mean value estimate

$\hat{E}(X^2)$ Ensemble-averaged mean value estimate of the process of the squares.

$\hat{E}_j$ Mean value estimate of *j*th record.

$\hat{E}(X_j^2)$ Mean value estimate of *j*th record of the process of the squares.

F_k Windowing factor for *k*th time lag.

$id_{1/3}$ Index corresponding to largest 1/3 of the peaks.

K_β One-half the non-dimensional width of the confidence interval.

M Index corresponding to the decorrelation time.

M_p Decorrelation index of the largest 1/3 of the peaks

N_a Total number of points.

Nc_j Number of dependent peaks in a group.

N_j Number of points in *j*th record.

N_G Number independent groups of peaks.

Nm_j Number of mean crossings in *j*th record.

N_p Total number of peaks.

N_s Number largest 1/3 of the peaks.

N_r Number of records.

PC Array of groups of dependent peaks.

Pc Value of a peak in the group of dependent peaks.

Pk Matrix of peak data.

Pks Matrix of sorted peak data.

Ps Matrix of the largest 1/3 of the peaks.

P_β Confidence probability.

Q Quantile of a distribution.

Q_N Quantile of standard normal distribution.

$\hat{R}_k(X)$ *k*th time lag of ensemble-averaged autocovariance function estimate of the process *X*.

$\hat{r}_k(X)$ *k*th time lag of ensemble-averaged autocorrelation function estimate of the process *X*.

$\hat{R}_k(X_j)$ *k*th time lag of autocovariance function estimate of *j*th record of the process *X*.

$\hat{R}_k(X^2)$ *k*th time lag of ensemble-averaged autocovariance function estimate of the process of the squares.

$\hat{R}_k(X_j^2)$ *k*th time lag of autocovariance function estimate of *j*th record of the process of the squares.

$\hat{R}_{a,k}^p$ *k*th time lag of averaged covariance function estimate of the largest 1/3 of the peaks.

$\hat{R}_{j,k}^p$ *k*th time lag of covariance function of the largest 1/3 of the peaks estimated over *j*th group of dependent peaks.

SSA Single significant amplitude.

T_a Total duration of the ensemble, s.

T_j Duration of the *j*th record, s.

TM Sample of mean-crossing time instances.

Tm Mean-crossing time instance, s.

t Time, s.

tp Time instant of mean-crossing peak, s.

$\mathbb{V}(\dots)$ Variance

$\hat{V}_a$ Ensemble-averaged variance estimate.

$\hat{V}_a^{x2}$ Ensemble-averaged variance estimate of the process of the squares.

$\hat{V}_j$ Variance estimate of of *j*th record.

$\hat{V}_j^{x2}$ Variance estimate of *j*th record of the process of the squares.

$\hat{\mathbb{V}}(\hat{E}_a)$ Variance of ensemble-averaged mean value estimate

$\widehat{\mathbb{V}}\left(\hat{E}(X^2)\right)$ Variance of ensemble-averaged mean value estimate of the process of squares

$\widehat{\mathbb{V}}(\hat{V}_a)$ Variance of ensemble-averaged variance estimate.

$\hat{V}_{\widehat{SSA}}$ Variance of SSA estimate.

$\hat{V}_{\widehat{SSA}}$ Variance of SSA estimate of *j*the record.

W_j Statistical weight of a record.

Wg_j Statistical weight of a group of dependent peaks.

X Data sample as an ensemble of records.

x Process value.

xp Absolute value of mean-crossing peak.

τ_{dc} Decorrelation time, s.

τ_m Time of the first minimum of the envelope of the autocorrelation estimate, s.

τ_s Time of crossing of the level of significance, s.

$\hat{\square}$ Estimate ("hat" above a symbol).

$\square_{Low}$ Lower boundary of confidence interval.

$\square_{Up}$ Upper boundary of confidence interval.

$\{\square_{j,i}\}$ Nested array: *i*th value of *j*th record.

8. REFERENCES

Belenky, V., V. Pipiras, C. Kent, M. Hughes, B. Campbell & T. Smith (2013) "On the Statistical Uncertainties of Time-domain-based Assessment of Stability Failures: Confidence Interval for the Mean and Variance of a Time Series." *Proc. 13th Int'l. Ship Stability Workshop,* Brest, France, pp. 251–258.

Belenky, V., V. Pipiras & K. Weems (2015) "Statistical Uncertainty of Ship Motion Data." Proc. 12th Int'l. Conf. Stability of Ships & Ocean Vehicles, Glasgow, UK, 12 p.

Belenky, V. & K. M. Weems (2019) "Dependence of Roll and Roll Rate in Nonlinear Ship Motions in Following and Stern Quartering Seas", Chapter 27 of Contemporary Ideas on Ship Stability. Risk of Capsizing, Belenky, V., K. Spyrou, F. van Walree, M. A. S Neves & N. Umeda, eds., Springer, ISBN 978-3-030-00514-6, pp. 455–473.

Bretschneider, C. L. (1959) "Wave Variability and Wave Spectra for Wind-Generated Gravity Waves," U.S. Army Beach Erosion Board, Tech. Memo. 118, Washington, DC, USA.

Glotzer, D., V. Pipiras, V. Belenky, K. Weems, & M. D. Levine (2023) "Statistical Inference for Mean and Variance of Oscillatory Processes." *Ocean Eng.*, Special Issue on Extreme Stochastic Dynamics of Ships, Vol. 289, 116215, 18 p.

ITTC (2005) "The Specialist Committee on Stability in Waves, Final Report and Recommendations to the 24th ITTC," Proc. 24th Int'l. Towing Tank Conf., Vol. II, pp. 369–407.

Levine, M. D., V. Belenky, K. Weems & V. Pipiras (2017) "Statistical Uncertainty Techniques for Analysis of Simulation and Model Test Data", Proc. 30th American Towing Tank Conf., West Bethesda, MD, USA.

Lobato, I. N. (2001), "Testing That a Dependent Process is Uncorrelated," J. American Statistical Ass'n, Vol. 96, pp. 1066–1076.

Pipiras, V., D. Glotzer, V. Belenky, M. Levine & K. Weems (2018) "On Confidence Intervals of Mean and Variance Estimates of Ship Motions" Proc. 13th Int'l. Conf. on Stability of Ships & Ocean Vehicles STAB 2018, Kobe, Japan.

Shin, Y. S, V. L. Belenky, W. M. Lin, K. M. Weems & A. H. Engle (2003) "Nonlinear time domain simulation technology for seakeeping and wave-load analysis for modern ship design" Trans. SNAME Vol. 111, pp. 557–578.

Spanos, D. & A. Papanikolaou (2009) "Benchmark Study on Numerical Simulation Methods for the Prediction of Parametric Roll of Ships in Waves," Proc. 10th Int'l. Conf. Stability of Ships & Ocean Vehicles, St. Petersburg, Russia, pp. 627–636.

Umeda, N., A. Munif & H. Hashimoto (2000) "Numerical Prediction of Extreme Motions and Capsizing for Intact Ships in Following/ Quartering Seas," Proc. 4th Osaka Colloquium on Seakeeping Performance of Ships, Osaka, Japan, pp. 368–373.

Weems, K. & V. Belenky (2023) "Reduced-order Model for Ship motions Incorporating a Volume-based Calculation of Body-nonlinear Hydrostatic and Froude-Krylov Forces", Ocean Eng., Special Issue on Extreme Stochastic Dynamics of Ships, Vol. 289, 116214, 14 p

Appendix A. DELTA METHOD FOR SSA

Eq. (28) is a deterministic function of a single random argument—the variance estimate $\hat{V}_X$. In the delta method, this function is expanded in a Taylor series around its mean value $V_0 = \mathbb{E}(\hat{V}_X)$

$$\widehat{SSA}(\hat{V}_X) \approx \widehat{SSA}(V_0) + \left.\frac{d\widehat{SSA}(\hat{V}_X)}{d\hat{V}_X}\right|_{\hat{V}=\hat{V}_0} (\hat{V}_X - V_0) + \frac{1}{2}\left.\frac{d^2\widehat{SSA}(\hat{V}_X)}{d\hat{V}_X^2}\right|_{\hat{V}=\hat{V}_0} (\hat{V}_X - V_0)^2 + \ldots \tag{A1}$$

Eq. (A1) is then linearized, so the terms of second order and higher are neglected. Evaluation of the derivative and substitution into Eq. (28) yields the following linearized function:

$$\widehat{SSA}_{Lin}(\hat{V}_X) = 2\sqrt{V_0} + \frac{(\hat{V}_X - V_0)}{\sqrt{V_0}}. \tag{A2}$$

The variance of the variance estimate is known: $\hat{V}_{\hat{V}_X}$ for each process of interest. The mean value of the estimate $V_0 = \mathbb{E}(\hat{V}_0)$ is a deterministic value, *i.e.* is a constant in Eq. (A1).

$$\mathbb{V}(\hat{V}_0) = 0 \quad \text{and} \quad \mathbb{V}(2\sqrt{V_0}) = 0.$$

Applying the rule for the variance of a linear function to Eq. (A2) yields:

$$\mathbb{V}\left(\widehat{SSA}_{Lin}(\hat{V}_x)\right) = \frac{\mathbb{V}(\hat{V}_x)}{V_0} \tag{A3}$$

The true mean value for the variance estimate $V_0 = \mathbb{E}(\hat{V}_x)$ is not known, so it is replaced by its estimate, which is the variance estimate itself:

$$V_0 \approx \hat{V}_X$$

and Eq. (A3) becomes:

$$\mathbb{V}\left(\widehat{SSA}_{Lin}(\hat{V}_x)\right) \approx \frac{\hat{V}_{\hat{V}_X}}{\hat{V}_x} = \hat{V}_{\widehat{SSA}}. \tag{A4}$$

Appendix B. CALCULATION EXAMPLE

B.1. General Description

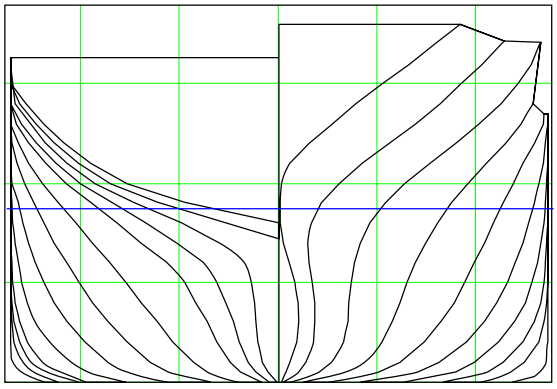

Figure B1: Body plan for the ITTC-A1 ship

An example for this procedure uses the ITTC-A1 ship (Umeda, et al., 2000), whose principal particulars are summarized in Table B1. This configuration was used in the ITTC parametric roll benchmarking (ITTC,

2005) and SAFEDOR project (*e.g.*, Spanos & Papanikolaou 2009). Roll decay data were available from the latter reference. The lines are shown in Figure B1.

A fast volume-based simulation tool (Weems & Belenky 2023) was used to generate sample data. The simulations included 3 degrees of freedom motions: heave-roll-pitch. A body-nonlinear formulation was applied for hydrostatic and Froude-Krylov forces. Diffraction and radiation for heave and pitch and diffraction for roll was approximated from the potential flow simulation tool LAMP (Large Amplitude Motion Program, Shin, et al. 2003) while added mass and damping for roll was extracted from roll decay test data from Spanos & Papanikolaou (2009).

Table B1: Hull form characteristics for the ITTC-A1 ship

Length BP, m	150
Breadth, m	27.2
Draft amidships, m	8.5
KG , m	10.24
GM ,m	1.38
C_B	0.667
C_M	0.959
C_{WP}	0.786

The wave environment was represented by long-crested irregular waves generated with a Bretschneider (1959) spectrum. The significant wave height was 7.5 m and the modal period was 14 s. The spectrum was discretized with 240 frequencies from 0.2 to 0.8 1/s. The time step was 0.5 s, with a ramp-up time of 10 s. Calculations were done for a forward speed of 10 knots in stern quartering seas corresponding to a relative heading angle of 45 degrees. Duration of a record without self-repeating effect was 40 min. In total, 300 records, totalling 200 hours of data, were produced.

B.2. Large Number of Records

For the illustration of calculations with the assumption of a large number of records available, a set of records with indices 100–130 was used, totaling 31 records covering 20.7 hours. Calculations are presented for roll motions. Estimates of mean value, variance, SSA and related values are shown in Table B2.

Table B2: Large number of records: mean and variance estimates

Estimate of mean value, deg	-0.833
Estimate of variance. deg^2	105.34
Variance of the mean, deg^2	$1.1\ 10^{-4}$
Variance of the variance, deg^4	3.452
Self-normalization: sum of squared cumulatives	
For mean value, deg^2	2.139
For variance, deg^4	$2.82\ 10^4$

Intermediate results for SSA calculations, using both the variance estimate and counting of peaks are given in Table B3. To evaluate decorrelation time, which is required for grouping the largest 1/3 of the peaks, the autocorrelation of motions was estimated; it is shown in Figure 3 of the main text of the procedure. The autocorrelation function of the largest 1/3 of the peaks is shown in Figure 6 of the main text of the procedure.

Table B3: Large number of records: SSA estimates

SSA from variance estimate, deg	20.52
Total number of peak values	9041
Number of largest 1/3 of the peaks	3014
Roll angle, corresponding to the largest 1/3 of the peaks, deg	15.53
SSA from averaging the largest 1/3 of the peaks, deg	20.6
Decorrelation time for the process, s	138.8
Number of groups of dependent peaks	125
Decorrelation index	5
Variance of SSA estimate, deg^2	0.02
Standard deviation of SSA estimate, deg	0.142
Sum of squared cumulatives for self-normalization, deg^2	2.847

Confidence intervals for mean value and variance estimates, constructed with normal distribution Eqs. (49) & (50) and self-normalization Eq. (57), are shown in Figures B2 and B3 respectively.

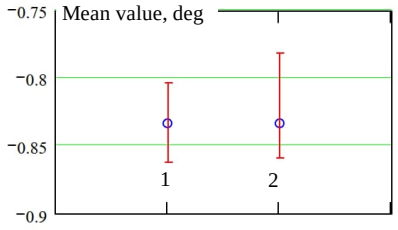

Figure B2: Confidence interval for mean values: (1) normal distribution (2) self normalization

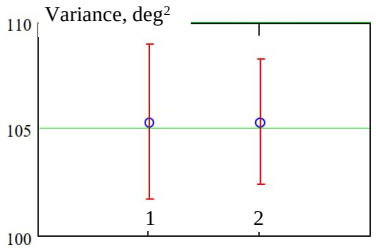

Figure B3: Confidence interval for variances: (1) normal distribution (2) self normalization

The confidence interval for the SSA was constructed in 5 different ways:

1. "Boundary" method Eq. (52) with SSA estimated using Eq. (28)
2. Delta method Eq. (53) with SSA estimated using Eq. (28)
3. "Boundary" method Eq. (52) applied to confidence interval constructed with self-normalization Eq. (57), with SSA estimated using Eq. (28)
4. SSA estimated by averaging the 1/3-highest peaks Eq. (37), confidence interval is constructed using normal distribution, while the variance of SSA is computed using Eq. (43)
5. SSA estimated by averaging the 1/3-highest peaks Eq. (37), confidence interval is constructed using self-normalization Eq. (57)

The results using these five methods are shown in Figure B4, where the numbers correspond to the methods given above.

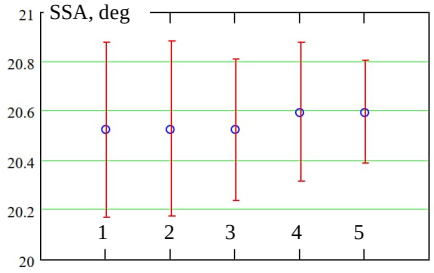

Figure B4: Confidence interval for SSA estimated using a large number of records, see the text for explanation of numbers

B.3. Any Number of Records

To illustrate calculations without the assumption of a large number of records available, a set of records with indices 1–7 was used, covering 4.7 hours in total. Calculations are again presented for roll motions. Estimates of mean value, variance, SSA and related values are given in Table B4. The ensemble autocorrelation function and the estimate of decorrelation time is depicted in Figure B5. Estimates of the autocovariance of squares and covariance of squares with the process values are shown in Figures B6 and B7, respectively.

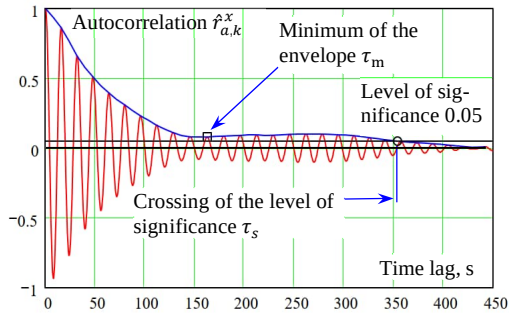

Figure B5: Estimate of decorrelation time with autocovariance function averaged over 7 records

Intermediate results of the SSA calculation, both for the variance estimate and using counting of peaks, are in Table B5, while the estimate of the autocorrelation function of the largest 1/3 of the peaks is shown in Figure B8.

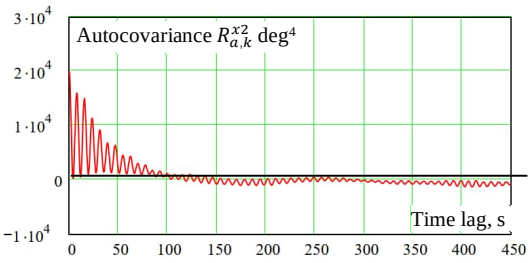

Figure B6: Estimate of autocovariance of squares of values of the process

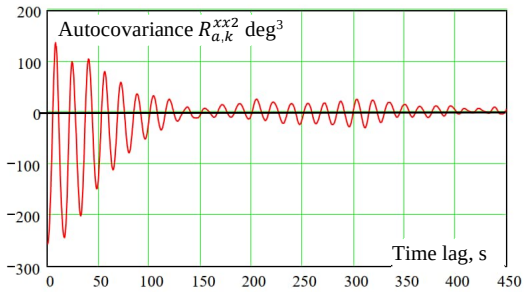

Figure B7: Estimate of autocovariance of squares and process values

Table B4: Mean and variance estimates using the method for any number of records.

Estimate of mean value, deg	-0.808
Estimate of variance. deg^2	99.62
Decorrelation time, s	162.5
Variance of the mean, deg^2	2.27 10^{-3}
Variance of the mean squares, deg^4	40.17
Covariance of the mean and mean squares, deg^3	-0.368
Variance of the variance, deg^4	38.99
Self-normalization: sum of squared cumulatives	
For mean value , deg^2	2.384
For variance, deg^4	6.45 10^4

Confidence intervals for the mean value and variance estimates, constructed with normal distribution Eqs. (49), (50) and self-normalization Eq. (57), are depicted in Figures B9 and B10 respectively.

Table B5: SSA estimates using the method for any number of records.

SSA from variance estimate, deg	19.96
Total number of peak values	2045
Number of largest 1/3 of the peaks	682
Roll angle, corresponding to the largest 1/3 of the peaks, deg	14.94
SSA from averaging the largest 1/3 of the peaks, deg	20.30
Number of groups of dependent peaks	29
Decorrelation index	5
Variance of SSA estimate, deg^2	0.068
Standard deviation of SSA estimate, deg	0.261
Sum of squared cumulatives for self-normalization, deg^2	4.86

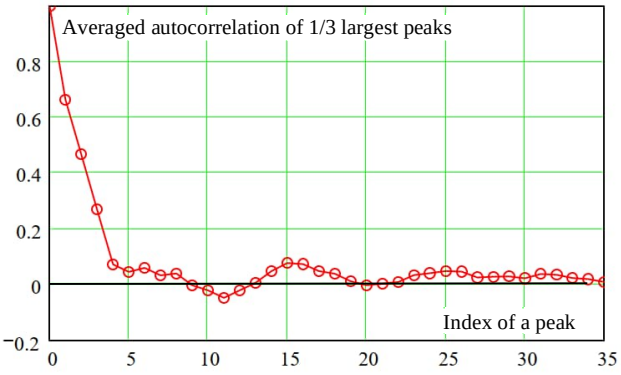

Figure B8: Estimate of averaged autocovariance for largest 1/3 of the peaks for roll motion

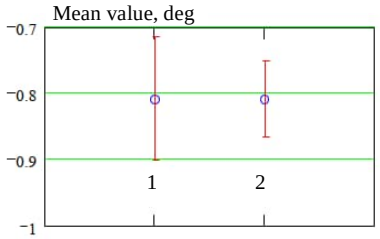

Figure B9: Confidence interval for mean values: (1) normal distribution (2) self normalization

The confidence interval for SSA was constructed in 5 different ways:

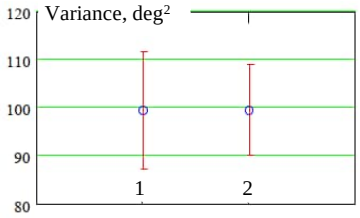

Figure B10: Confidence interval for variances: (1) normal distribution (2) self normalization

1. "Boundary" method Eq. (52) with SSA estimated using Eq. (28)
2. Delta method Eq. (53) with SSA estimated using Eq. (28)
3. "Boundary" method Eq. (52) applied to confidence interval constructed using self-normalization Eq. (57), with SSA estimated using Eq. (28)
4. SSA estimated by averaging the 1/3-highest peaks Eq. (37), confidence interval is constructed with normal distribution, while the variance of SSA is computed using Eq. (43)
5. SSA estimated by averaging the 1/3-highest peaks Eq. (37), confidence interval is constructed using self-normalization Eq. (57)

The results using these five methods are shown in Figure B11 where the numbers correspond to the methods given above.

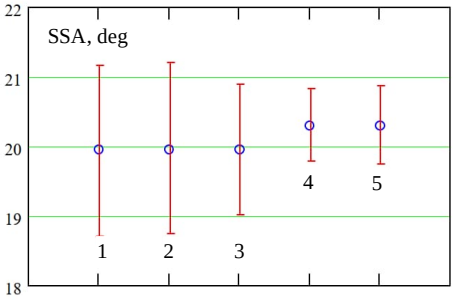

Figure B11: Confidence interval for SSA estimated using any number of records, see the text for explanation of numbers